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COEFFICIENT ESTIMATES FOR SPECIAL SUBCLASSES OF k -FOLD SYMMETRIC BI-UNIVALENT FUNCTIONS

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Abstract. In the present paper, we consider two new subclasses $\mathcal{N}_{\Sigma_k}(\mu, \alpha, \tau)$ and $\mathcal{N}_{\Sigma_k}(\mu, \beta, \tau)$ of Σ_k consisting of analytic and k -fold symmetric bi-univalent functions defined in the open unit disc $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. For functions belonging to the two classes introduced here, we derive their normalized forms. Furthermore, we find estimates of the initial coefficients $|a_{k+1}|$ and $|a_{2k+1}|$ for these functions. Several related classes are also considered and connections to previously known results are made.

1. INTRODUCTION

Let \mathcal{S} denote the family of functions analytic in the open unit disc

$$\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\},$$

and normalized by the conditions $f(0) = f'(0) - 1 = 0$ and having the form:

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j. \quad (1.1)$$

Also, let \mathcal{A} denote the subclass of functions in \mathcal{S} which are univalent in \mathcal{U} . The Koebe One Quarter Theorem (e.g., see [3]) ensures that the image of \mathcal{U} under every function $f(z) \in \mathcal{S}$ contains the disk of radius $\frac{1}{4}$. It is well known that every function f has an inverse f^{-1} satisfying:

$$f^{-1}(f(z)) = z, \quad (z \in \mathcal{U}) \text{ and } f(f^{-1}(w)) = w, \quad (|w| < r_0(f); r_0(f) \geq \tfrac{1}{4}),$$

where

$$f^{-1}(w) = g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots \quad (1.2)$$

A function $f \in \mathcal{S}$ is said to be bi-univalent in \mathcal{U} if both f and f^{-1} are univalent in \mathcal{U} . Let Σ denote the class of all bi-univalent functions in \mathcal{U} . Let Σ denote the class of all bi-univalent functions in \mathcal{U} . Examples of functions in class Σ are

$$h_1(z) = \frac{z}{1-z}, \quad h_2(z) = -\log(1-z), \quad h_3(z) = \frac{1}{2} \log\left(\frac{1+z}{1-z}\right), \quad z \in \mathcal{U}.$$

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For each function $f \in \mathcal{A}$, the function $h(z) = \sqrt[k]{f(z^k)}, (z \in \mathcal{U}, k \in \mathbb{N})$ is univalent and maps the unit disc \mathcal{U} into a region with k -fold symmetry. A function is said to be k -fold symmetric (see [7, 8]) if it has the following normalized form:

$$f(z) = z + \sum_{j=1}^{\infty} a_{kj+1} z^{kj+1}, \quad (z \in \mathcal{U}, k \in \mathbb{N}). \quad (1.3)$$

We denote \mathcal{S}_k the class of k -fold symmetric univalent functions in \mathcal{U} , which are normalized by the series expansion (1.3). In fact, the functions in the class \mathcal{A} are one-fold symmetric.

Analogously to the concept of k -fold symmetric univalent functions, their study gives some important results, such as the one saying that a function $f \in \Sigma$ generates a k -fold symmetric bi-univalent function for each $k \in \mathbb{N}$. Furthermore, for the normalized form of f given by (1.3), we obtain the series expansion for f^{-1} as follows:

$$\begin{aligned} g(w) = & w - a_{k+1} w^{k+1} + [(k+1)a_{k+1}^2 - a_{2k+1}] w^{2k+1} \\ & - \left[\frac{1}{2}(k+1)(3k+2)a_{k+1}^3 - (3k+2)a_{k+1}a_{2k+1} + a_{3k+1} \right] w^{3k+1} + \dots, \end{aligned} \quad (1.4)$$

where $f^{-1} = g$. We denote by Σ_k the class of k -fold symmetric bi-univalent functions in \mathcal{U} . For $k = 1$, the formula (1.4) coincides with the formula (1.2) of the class Σ .

Some examples of k -fold symmetric bi-univalent functions are given as follows:

$$\left(\frac{z^k}{1-z^k} \right)^{\frac{1}{k}}, \quad [-\log(1-z^k)]^{\frac{1}{k}}, \quad \left[\frac{1}{2} \log \left(\frac{1+z^k}{1-z^k} \right) \right]^{\frac{1}{k}}.$$

Recently, many authors investigated bounds for the various subclasses of k -fold symmetric bi-univalent functions (see [1, 2, 4, 10, 11, 13]). This work aims to introduce the new subclasses $\mathcal{N}_{\Sigma_k}(\mu, \alpha, \tau)$ and $\mathcal{N}_{\Sigma_k}(\mu, \beta, \tau)$ of Σ_k and find estimates of the coefficients $|a_{k+1}|$ and $|a_{2k+1}|$ for functions in each of these new subclasses.

2. MAIN RESULTS

Definition 2.1. A function $f \in \Sigma_k$ given by (1.3) is said to be in the class $\mathcal{N}_{\Sigma_k}(\mu, \alpha, \tau)$ if the following conditions are satisfied:

$$\left| \arg \left(1 + \frac{1}{\tau} \left[\frac{(1-\mu)(zf'(z) - f(z))}{(1-\mu)f(z) + \mu zf'(z)} \right] \right) \right| < \frac{\alpha\pi}{2} \quad (2.1)$$

and

$$\left| \arg \left(1 + \frac{1}{\tau} \left[\frac{(1-\mu)(wg'(w) - g(w))}{(1-\mu)g(w) + \mu wg'(w)} \right] \right) \right| < \frac{\alpha\pi}{2} \quad (2.2)$$

$$(0 < \alpha \leq 1; 0 \leq \mu < 1; \tau \in \mathbb{C} \setminus \{0\}; z, w \in \mathcal{U}),$$

where the function $g = f^{-1}$ is given by (1.4).

Definition 2.2. A function $f \in \Sigma_k$ given by (1.3) is said to be in the class $\mathcal{N}_{\Sigma_k}(\mu, \beta, \tau)$ if the following conditions are satisfied:

$$Re \left(1 + \frac{1}{\tau} \left[\frac{(1-\mu)(zf'(z) - f(z))}{(1-\mu)f(z) + \mu zf'(z)} \right] \right) > \beta \quad (2.3)$$

and

$$Re \left(1 + \frac{1}{\tau} \left[\frac{(1-\mu)(wg'(w) - g(w))}{(1-\mu)g(w) + \mu wg'(w)} \right] \right) > \beta \quad (2.4)$$

$$(0 \leq \beta < 1; 0 \leq \mu < 1; \tau \in \mathbb{C} \setminus \{0\}; z, w \in \mathcal{U}),$$

where the function $g = f^{-1}$ is given by (1.4).

Lemma 2.3. (See [6]) *If $h \in \mathcal{P}$, then $|c_k| \leq 2$ for each $k \in \mathbb{N}$, where \mathcal{P} is the family of all functions h analytic in \mathcal{U} , for which*

$$Re(h(z)) > 0, (z \in \mathcal{U})$$

where

$$h(z) = 1 + c_1 z + c_2 z^2 + \dots, (z \in \mathcal{U}).$$

Theorem 2.4. *Let $f \in \mathcal{N}_{\Sigma_k}(\mu, \alpha, \tau)$ ($0 < \alpha \leq 1; 0 \leq \mu < 1; \tau \in \mathbb{C} \setminus \{0\}$) be of the form (1.3). Then*

$$|a_{k+1}| \leq \frac{2\alpha|\tau|}{k(1-\mu)\sqrt{|1+\alpha(2\tau-1)|}} \quad (2.5)$$

and

$$|a_{2k+1}| \leq \frac{2\alpha^2|\tau|^2(k+1)}{k^2(1-\mu)^2} + \frac{\alpha|\tau|}{k(1-\mu)}.$$

Proof. It follows from (2.1) and (2.2) that

$$1 + \frac{1}{\tau} \left[\frac{(1-\mu)(zf'(z) - f(z))}{(1-\mu)f(z) + \mu zf'(z)} \right] = [p(z)]^\alpha \quad (2.6)$$

and

$$1 + \frac{1}{\tau} \left[\frac{(1-\mu)(wg'(w) - g(w))}{(1-\mu)g(w) + \mu wg'(w)} \right] = [q(w)]^\alpha, \quad (2.7)$$

where the functions $p(z)$ and $q(w)$ are in \mathcal{P} and have the following series representations:

$$p(z) = 1 + p_k z^k + p_{2k} z^{2k} + p_{3k} z^{3k} + \dots \quad (2.8)$$

and

$$q(w) = 1 + q_k w^k + q_{2k} w^{2k} + q_{3k} w^{3k} + \dots \quad (2.9)$$

Now, equating the coefficients in (2.6) and (2.7), we obtain

$$\frac{k(1-\mu)}{\tau} a_{k+1} = \alpha p_k, \quad (2.10)$$

$$\frac{(1-\mu)k}{\tau} [2a_{2k+1} - (1+k\mu)a_{k+1}^2] = \alpha p_{2k} + \frac{\alpha(\alpha-1)}{2} p_k^2, \quad (2.11)$$

and

$$-\frac{k(1-\mu)}{\tau} a_{k+1} = \alpha q_k, \quad (2.12)$$

$$\frac{(1-\mu)k}{\tau} [2(k+1)a_{k+1}^2 - 2a_{2k+1} - (1+k\mu)a_{k+1}^2] = \alpha q_{2k} + \frac{\alpha(\alpha-1)}{2} q_k^2. \quad (2.13)$$

From (2.10) and (2.12), we find

$$p_k = -q_k$$

and

$$2 \frac{k^2(1-\mu)^2 a_{k+1}^2}{\tau^2} = \alpha^2(p_k^2 + q_k^2). \quad (2.14)$$

From (2.11), (2.13) and (2.14), we get

$$\begin{aligned} \frac{2k^2(1-\mu)^2 a_{k+1}^2}{\tau} &= \alpha(p_{2k} + q_{2k}) + \frac{\alpha(\alpha-1)}{2} (p_k^2 + q_k^2) \\ &= \alpha(p_{2k} + q_{2k}) + \frac{(\alpha-1)k^2(1-\mu)^2}{\alpha\tau^2} a_{k+1}^2. \end{aligned}$$

Therefore, we have

$$a_{k+1}^2 = \frac{\alpha^2\tau^2(p_{2k} + q_{2k})}{k^2(1-\mu)^2 [1 + \alpha(2\tau-1)]}.$$

Applying Lemma 2.3, for the coefficients p_{2k} and q_{2k} , we have

$$|a_{k+1}| \leq \frac{2\alpha|\tau|}{k(1-\mu)\sqrt{|1 + \alpha(2\tau-1)|}}.$$

This gives the desired bound for $|a_{k+1}|$ as asserted in (2.5). In order to find the bound on $|a_{2k+1}|$, by subtracting (2.13) from (2.11), we get

$$\frac{2k(1-\mu)}{\tau} [2a_{2k+1} - (k+1)a_{k+1}^2] = \alpha(p_{2k} - q_{2k}) + \frac{\alpha(\alpha-1)}{2} (p_k^2 - q_k^2). \quad (2.15)$$

It follows from (2.14) and (2.15) that

$$a_{2k+1} = \frac{\alpha^2\tau^2(p_k^2 + q_k^2)(k+1)}{4k^2(1-\mu)^2} + \frac{\alpha\tau(p_{2k} - q_{2k})}{4k(1-\mu)}.$$

Applying Lemma 2.3 once again for the coefficients p_k , p_{2k} , q_k and q_{2k} , we readily obtain

$$|a_{2k+1}| \leq \frac{2\alpha^2|\tau|^2(k+1)}{k^2(1-\mu)^2} + \frac{\alpha|\tau|}{k(1-\mu)}.$$

The following theorem finds the estimates of the coefficients $|a_{k+1}|$ and $|a_{2k+1}|$ for functions in the class $\mathcal{N}_{\Sigma_k}(\mu, \beta, \tau)$. \square

Theorem 2.5. *Let $f \in \mathcal{N}_{\Sigma_k}(\mu, \beta, \tau)$ ($0 < \beta \leq 1$; $0 \leq \mu < 1$; $\tau \in \mathbb{C} \setminus \{0\}$) be of the form (1.3). Then*

$$\begin{aligned} |a_{k+1}| &\leq \frac{\sqrt{2|\tau|(1-\beta)}}{k(1-\mu)}, \\ |a_{2k+1}| &\leq \frac{|\tau|^2(1-\beta)^2(k+1)}{2k^2(1-\mu)^2} + \frac{|\tau|(1-\beta)}{k(1-\mu)}. \end{aligned} \quad (2.16)$$

Proof. It follows from (2.3) and (2.4) that

$$1 + \frac{1}{\tau} \left[\frac{(1-\mu)(zf'(z) - f(z))}{(1-\mu)f(z) + \mu zf'(z)} \right] = \beta + (1-\beta)p(z) \quad (2.17)$$

and

$$1 + \frac{1}{\tau} \left[\frac{(1-\mu)(wg'(w) - g(w))}{(1-\mu)g(w) + \mu wg'(w)} \right] = \beta + (1-\beta)q(w), \quad (2.18)$$

where $p(z)$ and $q(w)$ have the forms (2.8) and (2.9), respectively. By suitably comparing the coefficients in (2.17) and (2.18), we get

$$\left[\frac{k(1-\mu)}{\tau} \right] a_{k+1} = (1-\beta)p_k, \quad (2.19)$$

$$\frac{2k(1-\mu)a_{2k+1} - k(1-\mu)(1+k\mu)a_{k+1}^2}{\tau} = (1-\beta)p_{2k}, \quad (2.20)$$

and

$$-\left[\frac{k(1-\mu)}{\tau} \right] a_{k+1} = (1-\beta)q_k, \quad (2.21)$$

$$\frac{2k(1-\mu) [(k+1)a_{k+1}^2 - a_{2k+1}] - k(1-\mu)(1+k\mu)a_{k+1}^2}{\tau} = (1-\beta)q_{2k}. \quad (2.22)$$

From (2.19) and (2.21), we find

$$p_k = -q_k \quad (2.23)$$

and

$$\frac{2k^2(1-\mu)^2 a_{k+1}^2}{\tau^2} = (1-\beta)^2 (p_k^2 + q_k^2). \quad (2.24)$$

Adding (2.20) and (2.22), we have

$$\frac{2k^2(1-\mu)^2 a_{k+1}^2}{\tau} = (1-\beta) (p_{2k} + q_{2k}).$$

Applying Lemma 2.3, we obtain

$$|a_{k+1}| \leq \frac{\sqrt{2|\tau|(1-\beta)}}{k(1-\mu)}.$$

This is the bound on $|a_{k+1}|$ asserted in (2.16). In order to find the bound on $|a_{2k+1}|$, by subtracting (2.22) from (2.20), we get

$$\frac{2k(1-\mu) [2a_{2k+1} - (k+1)a_{k+1}^2]}{\tau} = (1-\beta) (p_{2k} - q_{2k}),$$

or equivalently,

$$a_{2k+1} = \frac{(k+1)a_{k+1}^2}{2} + \frac{\tau(1-\beta) (p_{2k} - q_{2k})}{4k(1-\mu)}.$$

It follows from (2.23) and (2.24) that

$$a_{2k+1} = \frac{\tau^2(1-\beta)^2(k+1) (p_k^2 + q_k^2)}{4k^2(1-\mu)^2} + \frac{\tau(1-\beta) (p_{2k} - q_{2k})}{4k(1-\mu)}.$$

Applying Lemma 2.3 once again for the coefficients p_k , p_{2k} , q_k and q_{2k} , we easily obtain

$$|a_{2k+1}| \leq \frac{|\tau|^2(1-\beta)^2(k+1)}{2k^2(1-\mu)^2} + \frac{|\tau|(1-\beta)}{k(1-\mu)}.$$

For one-fold symmetric bi-univalent functions and $\tau = 1$, Theorem 2.4 and Theorem 2.5 reduce to Corollary 2.8 and Corollary 2.9, respectively, which were proven very recently by Frasin [5] (see also [9]). \square

Definition 2.6. A function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{N}_\Sigma(\mu, \alpha)$ if the following conditions are satisfied:

$$\left| \arg \left(\frac{zf'(z)}{(1-\mu)f(z) + \mu zf'(z)} \right) \right| < \frac{\alpha\pi}{2}$$

and

$$\left| \arg \left(\frac{wg'(w)}{(1-\mu)g(w) + \mu wg'(w)} \right) \right| < \frac{\alpha\pi}{2}$$

$$(0 < \alpha \leq 1; 0 \leq \mu < 1; z, w \in \mathcal{U}),$$

where the function $g = f^{-1}$ is given by (1.2).

Definition 2.7. A function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{N}_\Sigma(\mu, \beta)$ if the following conditions are satisfied:

$$\operatorname{Re} \left(\frac{zf'(z)}{(1-\mu)f(z) + \mu zf'(z)} \right) > \beta$$

and

$$\operatorname{Re} \left(\frac{wg'(w)}{(1-\mu)g(w) + \mu wg'(w)} \right) > \beta$$

$$(0 \leq \beta < 1; 0 \leq \mu < 1; z, w \in \mathcal{U}),$$

where the function $g = f^{-1}$ is given by (1.2).

Corollary 2.8. Let $f \in \mathcal{N}_\Sigma(\mu, \alpha)$ ($0 < \alpha \leq 1; 0 \leq \mu < 1$) be of the form (1.1). Then

$$|a_2| \leq \frac{2\alpha}{(1-\mu)\sqrt{1+\alpha}}$$

and

$$|a_3| \leq \frac{4\alpha^2}{(1-\mu)^2} + \frac{\alpha}{1-\mu}.$$

Corollary 2.9. Let $f \in \mathcal{N}_\Sigma(\mu, \beta)$ ($0 \leq \beta < 1; 0 \leq \mu < 1$) be of the form (1.1). Then

$$|a_2| \leq \frac{\sqrt{2(1-\beta)}}{1-\mu}$$

and

$$|a_3| \leq \frac{(1-\beta)^2}{(1-\mu)^2} + \frac{1-\beta}{1-\mu}.$$

If we set $\mu = 0$ and $\tau = 1$ in Theorem 2.4 and Theorem 2.5, then the classes $\mathcal{N}_{\Sigma_k}(\mu, \alpha)$ and $\mathcal{N}_{\Sigma_k}(\mu, \beta)$ reduce to the classes $\mathcal{N}_{\Sigma_k}^\alpha$ and $\mathcal{N}_{\Sigma_k}^\beta$ investigated recently by Srivastava et al. ([12]).

Definition 2.10. A function $f \in \Sigma_k$ given by (1.3) is said to be in the class $\mathcal{N}_{\Sigma_k}^\alpha$ ($0 < \alpha \leq 1$) if the following conditions are satisfied:

$$\left| \arg \left(\frac{zf'(z)}{f(z)} \right) \right| < \frac{\alpha\pi}{2} \quad (z \in \mathcal{U})$$

and

$$\left| \arg \left(\frac{wg'(w)}{g(w)} \right) \right| < \frac{\alpha\pi}{2} \quad (w \in \mathcal{U})$$

and where the function g is given by (1.4).

Definition 2.11. A function $f \in \Sigma_k$ given by (1.3) is said to be in the class $\mathcal{N}_{\Sigma_k}^\beta$ ($0 \leq \beta < 1$) if the following conditions are satisfied:

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \beta \quad (z \in \mathcal{U})$$

and

$$\operatorname{Re} \left(\frac{wg'(w)}{g(w)} \right) > \beta \quad (w \in \mathcal{U}),$$

and where the function g is given by (1.4).

Corollary 2.12. Let $f \in \mathcal{N}_{\Sigma_k}^\alpha$ ($0 < \alpha \leq 1$) be of the form (1.3). Then

$$|a_{k+1}| \leq \frac{2\alpha}{k\sqrt{1+\alpha}}$$

and

$$|a_{2k+1}| \leq \frac{2\alpha^2(k+1)}{k^2} + \frac{\alpha}{k}.$$

Corollary 2.13. Let $f \in \mathcal{N}_{\Sigma_k}^\beta$ ($0 \leq \beta < 1$) be of the form (1.3). Then

$$|a_{k+1}| \leq \frac{\sqrt{2(1-\beta)}}{k}$$

and

$$|a_{2k+1}| \leq \frac{(1-\beta)^2(k+1)}{2k^2} + \frac{1-\beta}{k}.$$

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